A Problem

Find digits A,…,I such that

1. ABC + DEF = GHI,
2. A through I are distinct ,

and 3. Each of 1, 2, …, 9 is used exactly once.

218 + 349 = 567

How many solutions are there to this problem?

Possible Approaches

Count the number of possible sums

Count the number of possible addends

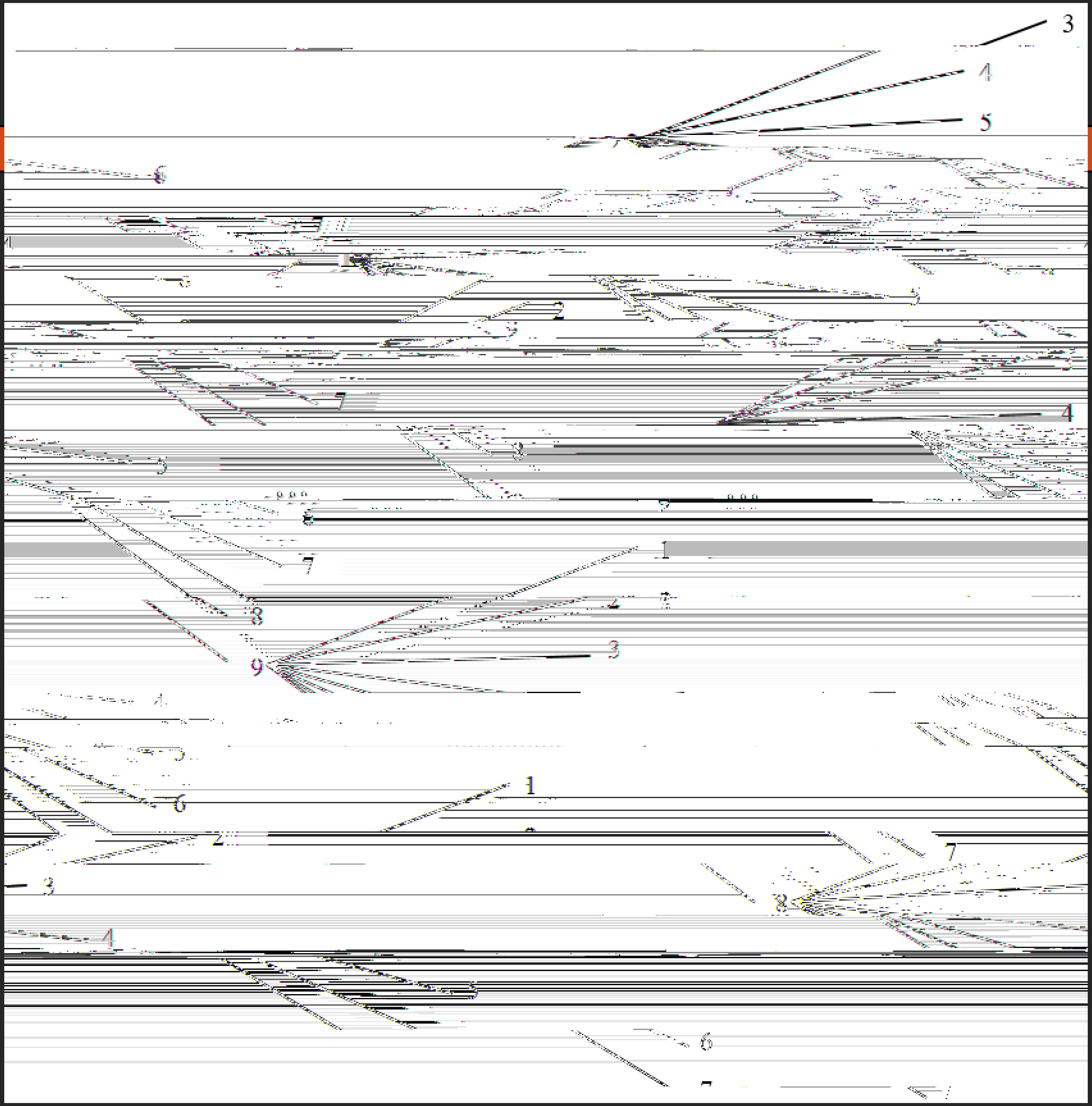
Count how many solutions there are for a given sum

218 + 349 = 567   219 + 348 = 567

248 + 319 = 567   249 + 318 = 567

A Little Counting

9(8)(7) = 504 3-digit numbers containing no 0’s such that all 3 digits are distinct.



A Little Counting

9(8)(7) = 504 3-digit numbers containing no 0’s such that all 3 digits are distinct.

But…the first digit of the sum cannot be 1 and cannot be 2. (Why not?)

What Else Can We Say?

The first digit cannot be 3 either:

$$\begin{array}{r} 1BC \\ +2EF \\ \hline 3HI \end{array}$$

$B + E < 10$

So, $\{B, E\} = \{4, 5\}$ and $H = 9$.

$\{C,F,I\} = \{6,7,8\}$ is impossible.

What Else Can We Say?

6(8)(7) = 336 3-digit numbers containing no 0's such that all 3 digits are distinct and the first digit is at least 4.

What else can we say?

Some Modular Arithmetic

We can use   to help.

Think remainders:

“7 is congruent to 1 mod 3”

means

$7 \div 3$ has a remainder of 1

Some Modular Arithmetic

We can use to help.

Think remainders:

"8 is congruent to __2__ mod 3"

means

8 ÷ 3 has a remainder of 2

Some Modular Arithmetic

We can use to help.

Think remainders:

"24 is congruent to __0__ mod 3"

means

24 ÷ 3 has a remainder of 0

Some Modular Arithmetic

We can use   to help.

Think remainders:

Any integer will be congruent to either
0, 1, or 2 mod 3.

Some Modular Arithmetic

How can we do arithmetic "mod 3?"

What is 7 + 10 congruent to mod 3?

7   1 (mod 3)

10   1 (mod 3)

So, 7 + 10   1 + 1 (mod 3).

Some Modular Arithmetic

How can we do arithmetic “mod 3?”

What is 7(10) congruent to mod 3?

7   1 (mod 3)

10   1 (mod 3)

So, 7(10)   1(1) (mod 3).

Back to the Problem

ABC + DEF = GHI

$$100A + 10B + C + 100D + 10E + F = 100G + 10H + I$$

$$A + B + C + D + E + F \quad G + H + I \pmod 3$$

Also,

$$A + B + C + D + E + F + G + H + I = 1 + \ldots + 9 = 45 \quad 0 \pmod 3$$

What could $A + \ldots + F$ and $G + H + I$ be congruent to mod 3?

1: $A + \ldots + F + G + H + I$

More Modular Arithmetic

$100A + 10B + C + 100D + 10E + F = 100G + 10H + I$

$A - B + C + D - E + F \quad G - H + I \pmod{11}$

$A + C + D + F + H \quad B + E + G + I \pmod{11}$

$A + \ldots + I \quad 2(B + E + G + I) \pmod{11}$

$2(B + E + G + I) \quad 1 \pmod{11}$

? $B + E + G + I \quad 6 \pmod{11}$

More Modular Arithmetic

$$\begin{array}{r} ABC \\ +DEF \\ \hline GHI \end{array}$$

$B + E + G + I \quad 6 \pmod{11}$

Look at B, E, and H:

We will have different cases based on whether we need to carry from the ones place or to the hundreds place.

More Modular Arithmetic

$B + E + G + I \quad\quad 6 \pmod{11}$

$$\begin{array}{r} ABC \\ +DEF \\ \hline GHI \end{array}$$

Look at B, E, and H:

B, E, H relationship	Carry?	
B + E = H	None	

More Modular Arithmetic

B + E + G + I

More Modular Arithmetic

$$B + E + G + I \qquad 6 \pmod{11}$$

Look at B, E, and H:

$$\begin{array}{r} ABC \\ +DEF \\ \hline GHI \end{array}$$

B, E, H relationship	Carry?	
B + E = H	None	
B + E + 1 = H	From Ones Place Only	
B + E = H + 10	To Hundreds Place Only	

More Modular Arithmetic

$$\begin{array}{r} ABC \\ +DEF \\ \hline GHI \end{array}$$

B + E + G + I 6 (mod 11)

Look at B, E, and H:

B, E, H relationship	Carry?	
B + E = H	None	

More Modular Arithmetic

$$\begin{array}{r} ABC \\ +DEF \\ \hline GHI \end{array}$$

B + E + G + I   6 (mod 11)

Look at B, E, and H:

B, E, H relationship	Carry?	Value of G + H + I
B + E = H	None	6 (mod 11)
B + E + 1 = H	From Ones Place Only	7 (mod 11)
B + E = H + 10	None	

Back to the Problem

412 GHI 987

7 G + H + I 24

and G + H + I is a multiple of 3.

9, 12, 15, $\boxed{18}$, 21, 24

Also, G + H + I must be congruent to 6, 7, or 8 mod 11.

So, G + H + I = 18.

Possible Sums

459	468	486	495	549	567	576
594	639	648	657	675	684	693
729	738	756	765	783	792	819
837	846	864	873	891	918	927
936	945	954	963	972	981	

(Three of these don’t work. Which ones?)

981

235 + 746 = 981

236 + 745 = 981

245 + 736 = 981

246 + 735 = 981

324 + 657 = 981

327 + 654 = 981

354 + 627 = 981

357 + 624 = 981

How many solutions are there in all?

What are they?

Try it!

Thank you!

jwaustin@salisbury.edu